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Polynomials with Positive Coefficients: Uniqueness of Best Approximation

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1. INTRODUCTION

In a recent paper [1] Chalmers has studied in a general framework the question of uniqueness of best approximation of a continuous function by polynomials which satisfy certain linear restrictions. His results are applicable to many of the standard constraints which have been investigated, such as monotone approximation [6], restricted range approximation [9, 10], restricted derivative approximation [8], and approximation by polynomials with bounded coefficients [7]. In all of these cases the uniqueness results had been demonstrated previously. The purpose of this note is to apply Chalmers' method to a situation in which uniqueness has not yet been established, and, thereby, to furnish an additional example of the utility of Chalmers' approach.

2. STATEMENT OF PROBLEM

Let V^n be the set of all algebraic polynomials of degree less than or equal to n, and let $V_0^n = \{p : p(x) = \sum_{k=0}^{n} a_k x^k (1-x)^{n-k}, a_k \geq 0, k = 0, 1, \ldots, n\}$. $p \in V_0^n$ is called a *polynomial with positive coefficients* (PPC). Such polynomials, which are generalizations of Bernstein polynomials, were studied by Jurkat and Lorentz [2] and Lorentz [4, 5], who were primarily concerned with density and degree of approximation questions. For f a nonnegative function in $C[0, 1]$ we consider the approximation of f by polynomials in V_0^n. For n fixed, it follows from the usual compactness arguments that there exists a best nth degree PPC approximation to f; that is, there exists $p^* \in V_0^n$ such that $\|f - p^*\| \leq \|f - p\|$ for all $p \in V_0^n$. Our concern is to demonstrate the uniqueness of p^*.

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Remark. Since each $p \in V_0^n$ is nonnegative, the restriction of nonnegativity imposed on f is a natural one. Indeed, there are simple examples of functions which are not nonnegative for which the best PPC approximation is not unique.

The notion of Hermite–Birkhoff interpolation (see [3] for definitions) has been crucial in uniqueness questions of this type. Here, however, there is a difference from the usual case, since we will have to consider Hermite–Birkhoff interpolation with linear combinations of derivatives prescribed. Problems of this type have been studied in [3].

3. The Main Results

THEOREM 1. *Let $f \in C[0, 1]$, $f(x) \geqslant 0$ for all $x \in [0, 1]$. Then there exists a unique $p^* \in V_0^n$ such that $\|f - p^*\| \leqslant \|f - p\|$ for all $p \in V_0^n$.*

Proof. If $p(x) = \sum_{k=0}^{n} a_k x^k (1-x)^{n-k}$, then, for $j = 0, 1,..., n$,

$$p^{(j)}(0) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i! \frac{(n-i)!}{(n-j)!} a_i = \sum_{i=0}^{j} b_{ji} a_i . \tag{1}$$

If we consider (1) as a system of linear equations in the unknowns a_i, then the matrix $B = (b_{ji})$ is triangular, with nonzero diagonal elements. Hence, B is nonsingular, so that there exists a unique solution to (1) given by

$$a_k = \sum_{j=0}^{k} \frac{1}{j!} \binom{n-j}{k-j} p^{(j)}(0), \qquad k = 0, 1,..., n. \tag{2}$$

We now define $n+1$ linear functionals on V_n by $L_k p = \sum_{j=0}^{k} c_{kj} p^{(j)}(0)$, $k = 0, 1,..., n$, where $c_{kj} = (1/j!)\binom{n-j}{k-j}$. Our linear constraints are

$$L_k p \geqslant 0, \qquad k = 0, 1,..., n, \tag{3}$$

since if p satisfies (3), then, by (2), $p \in V_0^n$.

We now use the results of [1, Example 4]. Let e_x denote point evaluation at x. To prove uniqueness, we must show that the set $S = \{L_{k_0}, L_{k_1},..., L_{k_r}, e_{x_{r+1}},..., e_{x_n}\}$ is independent in the dual of V^n for any $0 \leqslant k_0 < k_1 < \cdots < k_r \leqslant n$, $0 \leqslant x_{r+1} < \cdots < x_n \leqslant 1$, with $e_{x_j} \neq L_{k_i}$ for all $i = 1, 2,..., r$; $j = r+1,..., n$. (Note that $L_0 p = p(0)$ and $L_n p = p(1)$, and that these are the only point evaluations among the L_{k_i}. Thus, the restrictions $e_{x_j} \neq L_{k_i}$ may be replaced by $x_{r+1} \neq 0$ if $k_0 = 0$, and $x_n \neq 1$ if $k_r = n$. On the other hand, *if* $x_{r+1} = 0$ or $x_n = 1$, then we may replace $e_{x_{r+1}}$ by L_0 or e_{x_n} by L_n. Without loss of generality, we may thus assume that $x_{r+1} \neq 1$ and $x_n \neq 1$.) The independence of S is equivalent to the poisedness of the following Hermite–Birkhoff interpolation problem:

Let $0 < x_{r+1} < \cdots < x_n < 1$. Does there exist a nontrivial $p \in V^n$ such that

$$\sum_{j=0}^{k} c_{k,j} p^{(j)}(0) = 0, \qquad k = k_0, k_1, \ldots, k_r, \tag{4}$$

$$p(x_j) = 0, \qquad j = r+1, \ldots, n? \tag{5}$$

The answer to this question is given by the next result. A set of functions $\{u_i\}$, $i = 0, 1, \ldots, m$, is a *Chebyshev system* on (a, b) if every nontrivial linear combination of $\{u_i\}$ has at most m distinct zeros on (a, b).

LEMMA. *Let $0 \leq k_0 < k_1 < \cdots < k_m \leq n$ be a sequence of integers. Then the set of functions $\{x^{k_i}(1-x)^{n-k_i}\}$, $i = 0, 1, \ldots, m$ is a Chebyshev system on* $(0, 1)$.

Proof. Let $p(x) = \sum_{i=0}^{m} b_i x^{k_i}(1-x)^{n-k_i} = x^n \sum_{i=0}^{m} b_i((1-x)/x)^{n-k_i} = x^n q(x)$. Let $u = (1-x)/x$. Then $u \in (0, \infty)$ and $q(x) = t(u) = \sum_{i=0}^{m} b_i u^{n-k_i}$. Since $\{u^{n-k_i}\}$, $i = 0, 1, \ldots, m$, is a Chebyshev system on $(0, \infty)$ [11, p. 27], t has at most m distinct zeros on $(0, \infty)$ and q (and hence p) has at most m distinct zeros on $(0, 1)$.

To prove Theorem 1, suppose $p \in V^n$ satisfies (4) and (5). Writing $p(x) = \sum_{k=0}^{n} a_k x^k (1-x)^{n-k}$, we have from (4) that $a_k = 0$ for $k = k_0, k_1, \ldots, k_r$, so that $p(x) = \sum_{k=0, k \neq k_i}^{n} a_k x^k(1-x)^{n-k}$. p is thus a linear combination of $n - r$ functions of the form $x^k(1-x)^{n-k}$. Therefore, by the Lemma if p is nontrivial, then it cannot vanish at more than $n - r - 1$ points of $(0, 1)$. Hence, by (5), $p(x) \equiv 0$, so that S is independent, and p^* is unique.

We turn now to a characterization of the best PPC approximation. A point $x \in (0, 1)$ is called an *extreme point* for f, p if $|f(x) - p(x)| = \|f - p\|$. If $|f(0) - p(0)| < \|f - p\|$ and $L_i p = 0$ for $i = i_1, \ldots, i_m$, then 0 is said to be an extreme point of multiplicity m for f, p. If $|f(0) - p(0)| = \|f - p\|$ and $L_i p = 0$ for $i = i_1, \ldots, i_m$, then 0 is an extreme point of multiplicity m for f, p if $i_1 = 0$, and of multiplicity $m + 1$ if $i_1 > 0$. Finally, $x = 1$ is an extreme point for f, p if $|f(1) - p(1)| = \|f - p\|$ and $L_n p = 0$. The set E of all extreme points for a pair f, p is called the *extremal set*, and the number of points in E (counting multiplicity) is called the *order* of E. Our final result, which gives a partial characterization of the best PPC approximation, then follows from [1, Theorem 2].

THEOREM 2. *If $p^* \in V_0^n$ is the best* PPC *approximation to f, then there exists an extremal set for f, p^* of order $\geq n + 2$.*

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